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COMPETITIVE STABILITY UNDER WEAK GROSS SUBSTITUTABILITY:
THE "EUCLIDEAN DISTANCE" APPROACH

BY

KENNETH J. ARROW AND LEONID HURWICZ

TECHNICAL REPORT NO. 77
JULY 31, 1959

INSTITUTE FOR MATHEMATICAL STUDIES IN SOCIAL SCIENCES
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1.   Introduction.

1.1. In earlier papers [3], [1], the global stability of the competitive
equilibrium was investigated under different assumptions on the excess demand
functions. For the most part, the dynamic assumption was that the price of
each commodity moved proportionately to its excess demand; there may or may
not be a commodity distinguished as numéraire whose price is held fixed at 1.

Let there be $m+1$ commodities, numbered $0, \cdots, m$; the numéraire, if
any, is commodity 0 . Let P_k be the price of commodity k , P the vector
with components P_k $(k = 0, \cdots, m)$, and p the vector with components P_j
$(j = 1, \cdots, m)$; the components of p are sometimes also denoted by p_j . Let
$F_k(P)$ be the excess demand for commodity k at price vector P ; we assume,

(W) $P \cdot F(P) = 0$,

and for each $k = 0, 1, \cdots, m$;

(H) $F_k(P)$ is homogeneous of degree 0 ;

(C) $F_k(P)$ is continuous;

(B) $F_k(P)$ is bounded from below.

Assumption (H) is usual and follows from utility maximization subject
to the budget constraint.

The assumption (C) of continuity is to be understood here in an extended sense which permits positive infinite values of demand (negative infinite values are excluded by (B). If $F_j(P)$ is finite at some P, then continuity has the usual meaning; if infinite, then continuity means that

$$\lim_{n \to \infty} F_j(P^n) = + \infty \ ,$$

for any sequence $\{P^n\}$ converging to P.[1/] It may be remarked that, because of assumptions (W) and (B), $F_j(P)$ can only be infinite when $P_j = 0$. Admitting infinite excess demands for free goods does not seem unreasonable; however, if the reader prefers to insist on continuity in the stricter sense which requires finite-valuedness, all the results of this paper will, of course, be a fortiori true.[2/]

Assumption (W) is Walras' Law.

Assumption (B) is reasonable for the case of pure trade, in which case the maximum excess supply, which an individual can offer, is his initial holdings (see also [2], Section 2).

1.2. If (a) there is no numéraire, the dynamic system is

[1/] Cf. [3], Theorem 7 and footnote 36; also [2], Section 2.

[2/] It should be noted that the excess demand function resulting from maximizing any utility function with the frequently-assumed property of positive marginal utilities (e.g., $\sum_{k=0}^{m} \alpha_k \log X_k$, $\alpha_k > 0$) will tend to infinity as the corresponding price tends to zero; the function is continuous in the extended sense used here, but is not bounded as price varies over the unit simplex.

$$\dot{P}_k = 0 \ \text{if} \ P_k = 0 , \quad F_k(P) < 0 ,$$

(1)

$$= F_k(P) \ \text{otherwise} \quad (k = 0, \cdots, m) .$$

If (b) there is a numéraire, let $f_j(p) = F_j(1,p)$ be the excess demand for commodity j as a function of $p_1, \cdots, p_m$, when $P_0 = 1$. The dynamic system in this case is

(2)

$$dp_j/dt = 0 \ \text{if} \ p_j = 0 , \quad f_j(p) < 0 ,$$

$$= f_j(p) \qquad (j = 1, \cdots, m) .$$

In both cases, the price of each commodity (other than numéraire) varies proportionately to excess demand with an exception to prevent prices from becoming negative. By choice of suitable units of measurement, we may assume that the rate of change of prices equals excess demand; i.e., there is no loss of generality in choosing all proportionality coefficients equal to unity.

The dynamic system (1) will be referred to as the non-numéraire system, (2) as the numéraire system (the less descriptive terms, "non-normalized," and "normalized," respectively, were used in [1], Section 2).

1.3. In [3] and [1] considerable attention was given to the case where the commodities are all gross substitutes, that is $\partial F_j / \partial P_k > 0$, for all $j \neq k$. It was shown that both the numéraire and non-numéraire systems were stable in the (rather strong) sense that, beginning with any starting point, the solution of the differential equations (1) or (2) converged to the

(unique) equilibrium point (see [1], Section 4). Two methods of proof were given; in one, the "maximum norm" method, it was shown that the expression,

$$\max_{k} \; |(P_k / \bar{P}_k) - 1| \; ,$$

where $\bar{P}$ is any equilibrium point, was necessarily decreasing along any solution of (1) or (2) which did not start from an equilibrium point. In the second, the "Euclidean distance" approach, it was shown similarly that the square of the distance to any equilibrium,

$$\sum_{k=0}^{m} (P_k - \bar{P}_k)^2 \; ,$$

was decreasing along solutions of (1) and (2). The last fact follows from the statement that

(3) $\bar{P} \cdot F(P) > 0$,   provided $\bar{P}$ is equilibrium and P is not.

Statement (3) can, as shown in [3], p. 534, be interpreted as saying that Samuelson's Weak Axiom of Revealed Preference holds between any pair of price vectors of which exactly one is an equilibrium price vector.[3]

In subsequent unpublished work, Uzawa [6], McKenzie [4], and Morishima [5], extended these results to the case where the commodities are weak gross substitutes, that is, where it is only assumed that

(S) $\partial F_j/\partial P_k \geq 0$ for all $j \neq k$.

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[3] This is, of course, considerably weaker than assuming that the Weak Axiom of Revealed Preference holds for all pairs of price vectors.

All three made assumptions which implied the existence of an equilibrium price vector with all components positive. In this note we remove these restrictions by an extension of the "Euclidean distance" argument. That is, we show that (S) in conjunction with the other assumptions is sufficient to imply stability even if the equilibrium points have some zero components.

1.4. In the case of weak gross substitutes the equilibrium is not necessarily unique. As a result, there is more than one conceivable meaning of stability. Both Uzawa and McKenzie demonstrate the following property, called by Uzawa "quasi-stability" ([7], pp. 3-4): every solution (path) of the differential equations (1) or (2) is bounded and the distance from the moving point to the set of equilibria approaches zero. It does not follow from quasi-stability that the solution approaches any limit; for instance, it might oscillate in some way about the set of equilibrium points. The "Euclidean distance" method, however, establishes that in fact each solution does converge to a limit, which must, of course, be an equilibrium. Thus we establish the stability of the system in the sense used in [3], p. 524.[4]

1.5. In this paper we confine ourselves to the dynamic systems (1) and (2). In [1], Theorem 1, it was shown that the "maximum norm" argument demonstrated the stability of more general dynamic systems, in which the rate of change of prices may be, for example, a non-linear function of excess demand. The results of Uzawa and McKenzie apply to these more

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[4] Morishima's assumptions implied the uniqueness of equilibrium, so this question does not arise in his proof.

general systems. It is not known whether or not their theorems generalize
to the case where there does not exist at least one strictly positive equi-
librium vector; it can be shown that if equilibria with some zero components
are submitted, the solutions to the dynamic system which generalizes (1)
can be unbounded.

1.6. Our procedure is based on previous results which showed that (3)
implies both global stability and convexity of the set of equilibrium points
(see [2], Theorem 2). Hence, we need only demonstrate that (3) holds. We
first show that weak gross substitutability implies that the excess demand
for a free good must be independent of the prices of all other goods. Second,
we show that if there exists a positive equilibrium price vector, then (3)
holds in the case of weak gross substitutes; the method is a modification of
the argument used in [1] for the case of strong gross substitutes. From
these two results we can derive the validity of (3) (see Theorem 1 below).

In the argument use is made several times of a reduction of the commod-
ity space to a smaller number of dimensions in the following sense: if the
prices of a set S of commodities are put equal to zero $(P_k = 0$ for $k \in S)$
and if the excess demands for the free goods are bounded, then the system of
excess demand functions for the remaining commodities, $F_k(P)$ $(k \notin S)$, satis-
fy all the assumptions, (H), (C), (W), (B), and (S) , considered as functions
of the remaining prices, P_k $(k \notin S)$, and so any results proved for such
systems of functions can be applied to the reduced system (see Lemma 3).

2. Excess Demand for Free Goods.

For any vector X and any set of indices S , we will denote by X_S the vector with components X_j ($j \in S$) ; thus $P_S = (P_{J_1}, \cdots, P_{J_s})$, where $J_1, \cdots, J_s$ are the elements of S . The notation $F_S(P)$ is understood similarly. Also, $\tilde{S}$ is the complement of S with respect to the set of indices $(0, \cdots, m)$.

We first demonstrate:

Lemma 1. If (H), (C), (W), (B), and (S) hold, then there are constants F_j^O such that $F_j(P) = F_j^O$ for all F such that $P_j = 0$. Equivalently, under the assumption listed, if P' and P" are the price vectors such that $P'_j = P''_j = 0$, then $F_j(P') = F_j(P'')$.

Proof: Let P be any fixed vector such that $P_j = 0$, k any fixed index, $k \neq j$. Write

(4) $S = \{r : P_r = 0\}$, so that $P_S = 0$ always,

and

(5) $T = \{r : P_r > 0 , \quad r \neq k\}$.

First, suppose $P_k > 0$, and consider two subcases, according as T is or is not null. In the first case,

$$F_j(P) = F_j(P_S, P_k) = F_j(0, P_k) = F_j(0,1) ,$$

by (H) , so that F_j is independent of P_k . In the second case, write again with the aid of homogeneity,

$$F_j(P) = F_j(P_S, P_T, P_k) = F_j(0, P_T, P_k) = F_j(0, P_T/P_k, 1) .$$

If we differentiate with respect to P_k ,

$$\partial F_j / \partial P_k = \sum_{r \in T} (\partial F_j / \partial P_r)(-P_r/P_k^2) .$$

From (S) , the left-hand side is non-negative, the right-hand side non-positive, so that

$$\partial F_j / \partial P_k = 0 .$$

Thus F_j is independent of P_k for all P_k positive and, by continuity (C) of $F_j(P)$, for $P_k = 0$; since this holds for all $k \neq j$, the lemma is proved.

In the case of strict gross substitutability $(\partial F_j / \partial P_k > 0$ for $j \neq k)$, it was shown in [1], Lemma 1, that the excess demand for a free good is necessarily infinite, a special case of Lemma 1 of the present paper.

<u>Lemma 2</u>. If $F_j(P^1) \leq 0$ for some P^1 for which $P_j^1 = 0$, then $F_j(P) \leq 0$ for all P .

<u>Proof</u>: For any P , define Q so that

(6) $Q_j = 0 , \quad Q_k = P_k$ for all $k \neq j .$

From (H) and (S) it easily follows that $F_j(P)$ is a non-increasing function of P_j ; from (6),

$$(7) \qquad\qquad F_j(P) \leq F_j(Q) \ .$$

From Lemma 1, $F_j(Q) = F_j(P^1) \leq 0$, by hypothesis; the lemma follows from (7).

Lemma 3. If assumptions (H), (C), (W), (B), and (S) hold for a vector $F(P)$ of excess demand functions, and if, for some set S of indices, the inequality $F_{\widetilde{S}}(P) \leq 0$ holds for all P , then the vector of excess demand functions $F_S(P_S,0)$ also satisfies (H), (C), (W), (B), and (S) as functions of P_S ; further, if $\overline{P}_S$ is any equilibrium point for $F_S(P_S,0)$, then $(\overline{P}_S,0)$ is an equilibrium point for $F(P)$.

Proof: That assumptions (H), (C), (B), and (S) hold for $F_S(P_S,0)$ is obvious. Since (W) holds for $F(P)$, we can write

$$P \cdot F(P) = P_S \cdot F_S(P_S,P_{\widetilde{S}}) + P_{\widetilde{S}} \cdot F_{\widetilde{S}}(P_S,P_{\widetilde{S}}) = 0 , \quad \text{for all } P .$$

If we let $P_{\widetilde{S}}$ approach zero monotonically, it follows from (S) that F_S is monotone decreasing and therefore bounded by (B) . At the same time $F_{\widetilde{S}}$ is bounded from below by (B) , and from above by the hypothesis $F_{\widetilde{S}}(P) \leq 0$; hence, in the limit we have

$$P_S \cdot F_S(P_S,0) = 0 ,$$

which is the assertion (W) for the set of functions $F_S(P_S,0)$.

If $\bar{P}_S$ is an equilibrium for $F_S(P_S,0)$, then by definition

$$F_S(\bar{P}_S,0) \leq 0 .$$

Since $F_{\bar{S}}(\bar{P}_S,0) \leq 0$ by hypothesis, $F(\bar{P}_S,0) \leq 0$, so that $(\bar{P}_S,0)$ is an equilibrium by definition.

3. The "Revealed Preference" Relation Between Equilibrium and Disequilibrium Points.

Theorem 1. If $F(P)$ is a vector of excess demand functions satisfying (H), (C), (W), (B), and (S), then

$$\bar{P} \cdot F(P) > 0$$

provided $\bar{P}$ is an equilibrium price vector and P is not $(P \neq 0)$.

In Section 3.1 we prove the theorem for the case $\bar{P} > 0$; in 3.2 the general case is established.

3.1. The proof follows in general the lines of [1], Lemma 5. However, some modifications are needed: (1) In [1] the proof made essential use of the assumption of gross substitutability in the strict sense; as we shall see, this assumption is not needed.[5] In [1] it was assumed that the disequilibrium price vector was positive; again we shall find this assumption unnecessary.

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[5] Morishima [5] replaced the assumption of strict gross substitutability by one of indecomposability, but this is also unnecessary.

3.1.1. The magnitude $\bar{P} \cdot F(P)$ is measured in money and is independent of the units in which commodities are measured. Since (in 3.1) $\bar{P} > 0$, we may, without loss of generality, assume that

$$(8') \qquad \bar{P}_k = 1 \qquad (k = 0, \cdots, m) ,$$

so that

$$(8'') \qquad F(\bar{P}) = 0 .$$

For a given non-zero vector $P \equiv (P_0, P_1, \cdots, P_m)$ assume the commodities so numbered that

$$(9') \qquad P_k \geq P_{k+1} \qquad (k = 0, \cdots, m-1) ,$$

and hence, by $P \neq 0$,

$$(9'') \qquad P_0 > 0 .$$

Then define a sequence of vectors P^s $(s = 0, \cdots, m)$ by the conditions that

$$(10) \qquad P_k^s = \max(P_s, P_k) .$$

It may easily be seen from (9') and (10) that for $P^s = (P_0^s, P_1^s, \cdots, P_m^s)$

$$(11) \qquad P_k^s = \begin{cases} P_k & (k \leq s) , \\ P_s & (k > s) , \end{cases}$$

and

(12)
$$P_k^{s+1} = \begin{cases} P_k & (k \le s) , \\ P_{s+1} & (k > s) . \end{cases}$$

Thus the change from P^s to P^{s+1} consists in changing the last $m-s$ components from P_s to P_{s+1} . In more detail, we see that

$$P^0 = (P_0, \cdots, P_0) ,$$
$$P^1 = (P_0, P_1, P_1, \cdots, P_1) ,$$
$$\cdots \qquad \cdots$$
$$P^s = (P_0, \cdots, P_{s-1}, P_s, P_s, \cdots, P_s) ,$$
$$\cdots \qquad \cdots$$
$$P^m = (P_0, \cdots, P_{m-1}, P_m) .$$

From (8) we see that $P^0 = P_0 \bar{P}$; it follows from (H), (9"), and (8") that

(13)
$$F(P^0) = 0 .$$

Also, we note that

(14)
$$P^m = P .$$

The last $m-s$ components of P^{s+1} are, from (11), (12), and (9'), not greater than the corresponding components of P^s while the first $s+1$ components are the same. Hence, by substitutability (S)

(15)
$$F_k(P^{s+1}) \le F_k(P^s) \quad (k \le s) .$$

Suppose for the moment that $P_{s+1} > 0$; then from $(9')$ $P_s > 0$.
Write $Q^{s+1} = (P_s/P_{s+1})P^{s+1}$. Since by $(9')$ $P_s/P_{s+1} \geq 1$, we have
$Q_k^{s+1} \geq P_k^s$ $(k \leq s)$, $Q_k^{s+1} = P_k^s$ $(k > s)$ from (11) and (12). By (S)
$F_k(Q^{s+1}) \geq F_k(P^s)$ $(k > s)$; since by (H) $F_k(Q^{s+1}) = F_k(P^{s+1})$, we have

$$(16) \qquad F_k(P^{s+1}) \geq F_k(P^s) \quad (k > s) .$$

The assumption that $P_{s+1} > 0$ may now be dropped since by continuity (C) (16)
also holds if $P_{s+1} = 0$.

The inequality (16) can also be written in the form

$$F_k(P^{t+1}) \geq F_k(P^t) \quad (k > s \geq t \geq 0) .$$

By induction $F_k(P^{t+1}) \geq F_k(P^0)$ $(k > s \geq t \geq 0)$. But the right member of
the last inequality vanishes by (13), and if we set t equal to $s-1$, we
get

$$(17) \qquad F_k(P^s) \geq 0 \quad (k > s) .$$

3.1.2. In 3.1.3 below we shall use the results of 3.1.1 to establish
the following inequalities:

$$(18) \qquad \sum_{k=0}^{m} F_k(P^{s+1}) \geq \sum_{k=0}^{m} F_k(P^s) \quad (s = 0, 1, \cdots, m-1) ,$$

$$(19) \qquad \sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \quad \text{if } P^s \text{ is an equilibrium vector,}$$
P^{s+1} a disequilibrium vector,
$s = 0, 1, \cdots, m-1$.

From (18) it follows by induction that $\sum\limits_{k=0}^{m} F_k(P^s) \geq \sum\limits_{k=0}^{m} F_k(P^0)$ for all s . Set $s = m$ and recall (13), then

$$\sum_{k=0}^{m} F_k(P^m) \geq 0 .$$

From (8) and (14) we can write

(20) $\qquad\qquad \overline{P} \cdot F(P) \geq 0 \quad$ for all P .

If, in addition, we assume that $P = P^m$ is a disequilibrium vector, then, since P^0 is an equilibrium vector, there must be some s for which P^s is an equilibrium vector, P^{s+1} a disequilibrium vector. Then with the aid of (19) we have by the same argument

(21) $\qquad\qquad \overline{P} \cdot F(P) > 0 \quad$ for any disequilibrium P ,

which is the assertion of the theorem.

3.1.3. To prove (18) and (19), we consider **three cases**: (a) $P_s = P_{s+1}$; (b) $P_s > P_{s+1}$ and $F_{k'}(P^{s+1}) < F_{k'}(P^s)$ for some $k' \leq s$; (c) $P_s > P_{s+1}$ and $F_k(P^{s+1}) = F_k(P^s)$ for all $k \leq s$. (In view of (9) and (15), these cases are exhaustive.)

(a) In this case, from (10) $P^s = P^{s+1}$; hence

(22) $\qquad\qquad \sum\limits_{k=0}^{m} F_k(P^{s+1}) = \sum\limits_{k=0}^{m} F_k(P^s) \quad$ if $\quad P_s = P_{s+1}$.

(<u>b</u>) In this case we have

(23) $P_k, [F_k, (P^{s+1}) - F_k, (P^s)] < P_{s+1}[F_k, (P^{s+1}) - F_k, (P^s)]$,

since $P_k, \geq P_s > P_{s+1}$ by assumption and (9). From (15)

$$P_k[F_k(P^{s+1}) - F_k(P^s)] \leq P_{s+1}[F_k(P^{s+1}) - F_k(P^s)] \qquad (k \leq s) \ .$$

This, in conjunction with (23), yields

(24) $\displaystyle\sum_{k=0}^{s} P_k[F_k(P^{s+1}) - F_k(P^s)] < P_{s+1} \sum_{k=0}^{s} [F_k(P^{s+1}) - F_k(P^s)]$.

Now by (W) we have

$$\sum_{k=0}^{m} P_k^s \, F_k(P^s) = 0 = \sum_{k=0}^{m} P_k^{s+1} \, F_k(P^{s+1}) \ ,$$

and therefore, with the aid of (11) and (12) .d then (24),

(25) $\displaystyle 0 = \sum_{k=0}^{m} P_k^{s+1} \, F_k(P^{s+1}) - \sum_{k=0}^{m} P_k^s \, F_k(P^s) = \sum_{k=0}^{s} P_k[F_k(P^{s+1}) - F_k(P^s)]$

$\displaystyle \qquad + P_{s+1} \sum_{k=s+1}^{m} [F_k(P^{s+1}) - F_k(P^s)] + (P_{s+1} - P_s) \sum_{k=s+1}^{m} F_k(P^s)$

$\displaystyle \qquad < P_{s+1} \sum_{k=0}^{m} [F_k(P^{s+1}) - F_k(P^s)] + (P_{s+1} - P_s) \sum_{k=s+1}^{m} F_k(P^s)$

$\displaystyle \qquad \leq P_{s+1} \sum_{k=0}^{m} [F_k(P^{s+1}) - F_k(P^s)]$,

the last inequality following from (17) and the fact that $P_{s+1} - P_s < 0$.
From (25) we must have $P_{s+1} > 0$, and therefore (25) yields

(26) $\qquad \sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \quad$ if $\;\; P_{s+1} < P_s \;\;$ and $\;\; F_{k'}(P^{s+1}) < F_{k'}(P^s)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ for some $\;\; k' \leq s$.

$\qquad$ (c) In this case by assumption

$$\sum_{k=0}^{s} F_k(P^s) = \sum_{k=0}^{s} F_k(P^{s+1}) \; ,$$

so that from (16)

(27) $\qquad \sum_{k=0}^{m} F_k(P^{s+1}) \geq \sum_{k=0}^{m} F_k(P^s) \quad$ if $\;\; P_s > P_{s+1} \;\;$ and $\;\; F_k(P^{s+1}) = F_k(P^s)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ for all $\;\; k \leq s$.

$\qquad$ Suppose now that P^s is an equilibrium, P^{s+1} a disequilibrium vector.
Then $F(P^s) \leq 0$, so that from (15) $F_k(P^{s+1}) \leq 0 \;\; (k \leq s)$. Since P^{s+1} is
not an equilibrium, $F_k(P^{s+1}) > 0$ for some k so that

$$F_{k''}(P^{s+1}) > 0 \quad \text{for some} \quad k'' > s \; .$$

Since $F_{k''}(P^s) \leq 0$, $F_{k''}(P^{s+1}) > F_{k''}(P^s)$, therefore, because of (16),

(28) $\qquad \sum_{k=0}^{m} F_k(P^{s+1}) > \sum_{k=0}^{m} F_k(P^s) \quad$ if $\;\; P_s > P_{s+1} \;\;$ while $\;\; F_k(P^{s+1}) = F_k(P^s)$

$\qquad\qquad\qquad\qquad\qquad$ for all $\;\; k \leq s$, with P^s equilibrium, P^{s+1} disequilibrium.

Then (18) follows from (22), (26), and (27). If P^s is equilibrium, P^{s+1} is disequilibrium, then $P^s \neq P^{s+1}$ and therefore $P_s > P_{s+1}$; then (19) follows from (26) or (28). Hence, by the argument of Section 3.1.2 (20) and (21) are demonstrated.

3.1.4. <u>Remark</u>. Let P be any equilibrium point; then $F(P) \leq 0$. From (20) it follows that

(29) if there exists a positive equilibrium, then $F(P) = 0$ for every equilibrium price vector P .

3.2. We now consider the general case where the equilibrium $\bar{P}$ may have zero components. Write

$$(30) \qquad\qquad S = \{k : \bar{P}_k > 0\} .$$

Then $\bar{P}_{\tilde{S}} = 0$. By definition of equilibrium $F(\bar{P}) \leq 0$ and, in particular, $F_{\tilde{S}}(\bar{P}) \leq 0$; by Lemma 2

$$(31) \qquad\qquad F_{\tilde{S}}(P) \leq 0 \quad \text{for all} \quad P .$$

Then from (B) , $F_{\tilde{S}}(P)$ is finite-valued so that the product $\bar{P}_{\tilde{S}} \cdot F_{\tilde{S}}(P)$ is well defined and equal to zero. Hence,

$$(32) \qquad \bar{P} \cdot F(P) = \bar{P}_S \cdot F_S(P) + \bar{P}_{\tilde{S}} \cdot F_{\tilde{S}}(P) = \bar{P}_S \cdot F_S(P) .$$

Since $P_{\tilde{S}} \geq 0$, it follows from (S) that

(33) $\qquad F_S(P) = F_S(P_S, P_S) \geq F_S(P_S, 0)$,

so that

(34) $\qquad \overline{P}_S \cdot F_S(P) \geq \overline{P}_S \cdot F_S(P_S, 0)$.

From (31) and Lemma 3 the function $F_S(P_S, 0)$ satisfies all the assumptions (H), (C), (W), (B), and (S). Further it has a positive equilibrium $\overline{P}_S$. Hence, we can apply the results (21) and (29) of Section 3.1 and conclude that

(35) $\qquad \overline{P}_S \cdot F_S(P_S, 0) > 0$ if P_S is not an equilibrium of $F_S(P_S, 0)$,

(36) $\qquad F_S(P_S, 0) = 0$ if P_S is an equilibrium of $F_S(P_S, 0)$.

If P_S is not an equilibrium of $F_S(P_S, 0)$, then $\overline{P} \cdot F(P) > 0$ from (32), (34-35). If it is, then $F_S(P) \geq 0$ from (33) and (36). If $F_S(P) = 0$ then, from (31), P is an equilibrium. Hence, if P is a disequilibrium vector, $F_S(P) \geq 0$, $F_S(P) \neq 0$, so that

$$\overline{P}_S \cdot F_S(P) > 0 ,$$

and, by (32), $\overline{P} \cdot F(P) > 0$ even if P_S is an equilibrium of $F_S(P_S, 0)$.

4. Statement of Theorem on Stability and Convexity of Equilibria.

In [1], as part of the proof of Theorem 2, it was shown that the condition $\overline{P} \cdot F(P) > 0$ was sufficient for the convergence of solutions of both

the non-numéraire and numéraire systems; for further discussion see [2],
Theorem 2, where "corners" are explicitly treated, and it is also shown
that the same condition insures that the set of equilibria is convex.
Hence, Theorem 1 implies

Theorem 2. If $F(P)$ is a vector of excess demand functions which
are homogeneous of degree zero, continuous, bounded from below, if Walras'
Law holds, and if all commodities are weak gross substitutes, then both
the non-numéraire and numéraire adjustment systems (Equations (1) and (2)
above) are stable[6] and the set of equilibria is convex.[7]

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[6] In the sense that every solution (path) converges to some equilibrium
point.

[7] The convexity of the set of equilibria when all commodities are weak
gross substitutes was observed by McKenzie [4].

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